Supersymmetry and Topological Invariants

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Abstract

In this report we discuss the use of the Witten index as a criterion for predicting whether symmetry is unbroken in a supersymmetric theory. We then proceed to prove the Atiyah-Singer index theorem for the case of the exterior derivative operator on forms, on a manifold M by calculating the Witten index for the SUSY σ-model.

Introduction

Symmetries have played a very important role as guiding principles in Theoretical Physics. A given symmetry enables us to make some statement about the behaviour of a system (namely the associated conserved charge), thus one would naturally be interested in any possible new symmetry that physical models could have. However, there is a theorem due to Coleman and Mandula [1], that forbids the possibility of any symmetries that transform as tensors under Lorentz transformations other than P_μ and $M_{\mu\nu}$. There is a very simple plausibility argument by Witten [2] as to why this should be so. It basically says that, the conservation of P_μ and $M_{\mu\nu}$, leaves only the scattering angle unknown. Any new conservation law would leave us with only a discrete set of possible angles, but since the scattering amplitude is an analytic function of angle it must be zero for all angles. Thus it would seem that the Coleman-Mandula theorem, rules out all possible symmetries other than the Poincaré group.

1 Supersymmetry

The above discussion & the Coleman-Mandula theorem, applied to conserved quantities that transform as tensors under the Lorentz transformations. However, it does not rule out conserved charges which transform as spinors (Witten's argument that we advanced about additional constraints making the S-matrix trivial, would not apply, because, these conserved charges being spinors, do not impose any additional constraints as they don't give us any new observables. What they *do* is to relate scattering of bosons to the scattering of their superpartners)

Since, these generators themselves have spin-$\frac{1}{2}$, they relate bosons & fermions. Being spinors they would satisfy anticommutation relations rather than commutation relations among themselves. We have[1]:

$$\begin{aligned} \{\hat{Q}, \hat{Q}^\dagger\} &= \mathcal{H} \\ \{\hat{Q}, \hat{Q}\} = & \; 0 \; = \{\hat{Q}^\dagger, \hat{Q}^\dagger\} \end{aligned}$$

Now, from the algebra of the supersymmetry generators, we can see several interesting things. Firstly, the Hamiltonian is always non-negative, because:

$$\langle\psi|\hat{\mathcal{H}}|\psi\rangle = \langle\psi|\{\hat{Q}^\dagger, \hat{Q}\}|\psi\rangle = ||\hat{Q}|\psi\rangle||^2 + ||\hat{Q}^\dagger|\psi\rangle||^2 \geq 0$$

Secondly, if a state $|\psi\rangle$ is supersymmetric, then since it is annihilated by $\hat{Q}$ and $\hat{Q}^\dagger$, therefore, its energy is:

$$\langle\psi|\hat{\mathcal{H}}|\psi\rangle = ||\hat{Q}|\psi\rangle||^2 + ||\hat{Q}^\dagger|\psi\rangle||^2 = 0$$

[1]For a more detailed review of supersymmetry refer to [2], [3] or [4]

Thus, *if* a state is supersymmetric, its energy is necessarily 0, *and* it is therefore a ground state. Hence we see that vacua can be supersymmetric, iff they are at 0 energy.

2 The Witten Index $Tr(-1)^{\hat{F}}$

2.1 A Criterion for Dynamical Supersymmetry breaking

Given any system with a supersymmetric lagrangian, we are always interested in knowing whether or not the supersymmetry is spontaneously broken. This is made difficult by the fact that if we calculate the energy of the vacuum in perturbation theory and find it to be zero, there may always be higher order corrections which make it non-zero. Thus if the vacuum energy is ≥ 0 then we can be sure that supersymmetry is broken, but if it seems to be 0 upto finite orders in perturbation, we can never be sure if supersymmetry is indeed unbroken, or it is broken at some higher order in perturbation theory.

Consider the operator $(-1)^{\hat{F}}$, which acts as:

$$(-1)^{\hat{F}}|f\rangle = -|f\rangle$$
$$(-1)^{\hat{F}}|b\rangle = +|b\rangle$$

where $|f\rangle$ & $|b\rangle$ are fermionic and bosonic states respectively. Now the quantity $Tr(-1)^{\hat{F}}e^{-\beta\hat{\mathcal{H}}}$ is quite interesting. Every bosonic state contributes $+e^{-\beta\hat{E}}$ & every fermionic state contributes $-e^{-\beta\hat{E}}$ to $Tr(-1)^{\hat{F}}e^{-\beta\hat{E}}$. But for every bosonic(fermionic) state, $|\psi\rangle$, of non-zero energy, there is a corresponding fermionic(bosonic) state $\hat{Q}|\psi\rangle$ of the same energy with an equal and opposite contribution to $Tr(-1)^{\hat{F}}e^{-\beta\hat{\mathcal{H}}}$, so all these pairs cancell off, and we are left with

$$\Delta = Tr(-1)^{\hat{F}}e^{-\beta\hat{\mathcal{H}}} = n_B^{(0)} - n_F^{(0)}$$

Where $n_F^{(0)}(n_B^{(0)})$ is the number of fermionic(bosonic) ground states. The interesting thing that Witten observed ([5] & [6]) was that if we make a smooth change of parameters which changes the spectrum continuously. Then it shifts states up and down in energy, always in boson-fermion pairs. Therefore, since the change in $n_F^{(0)}$ & $n_B^{(0)}$ is always equal, $Tr(-1)^{\hat{F}}$ is invariant[2]. This is a remarkable observation, which says that we can calculate the Witten index Δ for any model by looking at some much simpler theory in a suitable limit of the parameters. If this turns out to be nonzero, then at least on of $n_F^{(0)}$ or $n_B^{(0)}$ is $\neq 0$ and hence supersymmetry is definitely not broken. Therefore, a nonzero Witten index is a sufficient condition to conclusively decide that supersymmetry in unbroken in the ground state.

Now, we can rewrite Δ using path integrals[1] as

$$\Delta = Tr(-1)^{\hat{F}}e^{-\beta\hat{\mathcal{H}}} = \int_{[PBC]} [d\phi] e^{-\int_0^\beta \mathfrak{L}_E(\phi,\partial_u)du}$$

Where $\mathfrak{L}_E(\phi,\partial_u)$ is the Eulclidean lagrangian, and the path integral is over all possible field configurations satisfying periodic boundary conditions. The reason we have periodic boundary conditions instead of antiperiodic boundary conditions is the operator $Tr(-1)^{\hat{F}}$, which contributes a minus sign to the path integral[3]

3 The Nonlinear σ-Model

Now we consider the Nonlinear σ-Model with a lagrangian given by[4]

$$\mathfrak{L}(\phi,\psi,i\partial_t) = \frac{1}{2}g_{ij}\dot{\phi^i}\dot{\phi^j} + ig_{ij}\psi^{i*}\frac{D}{dt}\psi^j + \frac{1}{4}R_{ijkl}\psi^{i*}\psi^{j*}\psi^k\psi^l$$

with the corresponding Euclidean lagrangian:

$$\mathfrak{L}_E(\phi,\psi,\partial_u) = -\mathfrak{L}(\phi,\psi,i\partial_t) = \frac{1}{2}g_{ij}\phi^{i\prime}\phi^{j\prime} + g_{ij}\psi^{i*}\frac{D}{du}\psi^j - \frac{1}{4}R_{ijkl}\psi^{i*}\psi^{j*}\psi^k\psi^l$$

[2] We are assuming here that under this change of parameters, there is not continuum of energy states that comes down to zero energy. If such a continuum comes down to zero energy, or if there was a continuum at zero energy in the original spectrum, then our argument breaks down, and the Witten index *need* not necessarily remain unchanged

[3] Refer to Appendix A for more details

[4] Refer to [7] for an introduction to the nonlinear σ-model

Where the bosonic fields ϕ^i are maps from $\Re \rightarrow M$, where M is some n dimensional manifold, and the fermions, ψ^i are vectors on M and $\frac{D}{dt}\psi^j$ is the covariant derivative associated with Γ^j_{mn}, the Levi-Civita connection on M

$$\frac{D}{dt}\psi^j = \frac{d}{dt}\psi^j + \Gamma^j_{mn}\psi^m \frac{d}{dt}\phi^n$$

Now in this quantum mechanical model, canonical quantisation of ψ gives the commutation relations

$$\{\psi^i, \psi^{j*}\} = g^{ij}, \qquad \{\psi^i, \psi^j\} = \{\psi^{i*}, \psi^{j*}\} = 0$$

The states of this system may be represented by elements in $\Lambda(M)$, the exterior algebra on M. This is the space of all differential forms, and fermionic(bosonic) states correspond to odd(even) forms. This is a natural correspondence, since any state of the system with p fermions, can be constructed by the action of p creation operators on the vacuum. Due to the anticommuting nature of the ψ^*s this state can be thought of as a p-form composed of the wedge product of p 1-forms. The is so because the supersymmetry generator, $\hat{Q}$ is realised by

$$Q = p_i \psi^{i*} = \frac{D}{D\phi^i}\psi^{i*}$$

So given a state $|\Omega\rangle$

$$\begin{aligned} |\Omega\rangle &= A_{i_1 i_2 \cdots i_p}\psi^{i_1 *}\psi^{i_2 *}\cdots\psi^{i_p *}|0\rangle \\ \hat{Q}|\Omega\rangle &= \frac{D}{D\phi^{i_0}}\psi^{i_0 *}A_{i_1 i_2 \cdots i_p}\psi^{i_1 *}\psi^{i_2 *}\cdots\psi^{i_p *}|0\rangle \\ &= (dA)_{i_0 i_1 \cdots i_p}\psi^{i_0 *}\psi^{i_1 *}\cdots\psi^{i_p *}|0\rangle \end{aligned}$$

The action of this on a state is the same as the action of the exterior derivative d on forms. Therefore, the supersymmetry charges $\hat{Q}$ and $\hat{Q}^\dagger$ correspond to d and d^*, the exterior derivative and its adjoint acting on forms[5].

Thus we can see that the Hamiltonian now corresponds to the laplacian acting on forms:

$$\hat{\mathcal{H}} = dd^* + d^*d$$

Now, the supersymmetric states are states of zero energy which correspond to harmonic forms. $\therefore$ the Witten index, Δ is:

$$\Delta = n_B^{(0)} - n_F^{(0)} = \sum_{k=even}^{n} b_k - \sum_{k=odd}^{n} b_k = \sum_{k=0}^{n}(-1)^k b_k$$

Where b_k is the k^{th} Betti number, which is the number of linearly independent k-forms which are closed but not exact[6]. This expression is actually equal to $\chi(M)$, the Euler characteristic of the manifold M, as we shall show shortly. This is remarkable because it connects the index of the operator[7] d on M to a purely topological property of M, namely its Euler characteristic.

Calculation of the Witten Index

We have,

$$\begin{aligned} \Delta &= \int_{[PBC]} \exp\left(-\int_0^\beta \mathfrak{L}_E(\phi, \psi, \partial_u)du\right) \\ &= \int_{[PBC]} [d\phi][d\psi][d\psi^*]\exp\left(-\int_0^\beta \mathfrak{L}_E(\phi, \psi, \partial_u)du\right) \end{aligned}$$

[5] The adjoint of d is the operator d^* which takes p-forms to $(p-1)$-forms. Suppose $A_{i_1 i_2 \cdots p}$ is a p-form, then the action of d^* on it is given by, $(d^*A)_{i_1 i_2 \cdots p-1} = -pD^{i'}A_{i' i_1 i_2 \ldots i_{p-1}}$. For a very nicely written brief review of differential forms refer to Ch. 12 of [12]

[6] This is because there is a one-to-one correspondence between harmonic p-forms and the p^{th} cohomology group of M, $\mathcal{H}^p(M, R)$.

[7] The (analytical) index of an operator $\mathcal{O}$ is defined by $Ind(\mathcal{O}) = dim\{ker(\mathcal{O}^*\mathcal{O})\} - dim\{ker(\mathcal{O}\mathcal{O}^*)\}$

Now, for the nonlinear σ-model the exponent is:

$$\int_0^\beta \mathfrak{L}_E(\phi,\psi,\partial_u)du = \int_0^\beta du\left(\frac{1}{2}g_{ij}\phi^{i\prime}\phi^{j\prime} + g_{ij}\psi^{i*}\frac{D}{du}\psi^j - \frac{1}{4}R_{ijkl}\psi^{i*}\psi^{j*}\psi^k\psi^l\right)$$

Expanding the fields in a Fourier series, with $\omega = 2\pi/\beta$ we have,

$$\phi^i = \sum_{p=-\infty}^{\infty}\phi_p^i e^{(ip\omega u)},\quad \psi^i = \sum_{p=-\infty}^{\infty}\psi_p^i e^{(ip\omega u)},\quad \psi^{i*} = \sum_{p=-\infty}^{\infty}{\psi_p^i}^* e^{(ip\omega u)}$$

Where the ϕ_n^i are just real numbers and ${\psi_n^i}^*$ & ψ_n^i are Grassman numbers. Therefore for the non-constant modes:

$$\begin{aligned}\int_0^\beta \mathfrak{L}_E du &= \sum_{p,q=-\infty}^{\infty}\int_0^\beta du\left(\frac{-pq\omega^2}{2}g_{ij}(\phi_0^i)\phi_p^i\phi_q^j e^{i(q+p)\omega u} + ip\omega g_{ij}(\phi_0^i){\psi_q^i}^*\psi_p^j e^{i(q+p)\omega u} + \dots\right)\\ &= \sum_{p=-\infty}^{\infty}\beta\left(\frac{p^2\omega^2}{2}g_{ij}(\phi_0^i)\phi_p^i\phi_{-p}^j + iq\omega g_{ij}(\phi_0^i){\psi_{-p}^i}^*\psi_p^j + \dots \text{ higher order terms}\right)\end{aligned}$$

The reason that the higher order terms were neglected is that, as we saw earlier Δ is independent of β. Therefore, we can set β to any value in the path integral to make calculations easier. Making β arbitrarily small, will increase ω and thus suppress all non-constant modes. The contribution of these non-constant modes to the path integral is:

$$\begin{aligned}\mathcal{I}_1 &= \int\prod_{p\neq 0}\left(\sqrt{\det(g)}\prod_{i=1}^n\frac{d\phi_p^i}{\sqrt{2\pi}}\right)\left(\prod_{j=1}^n d{\psi_p^j}^* d\psi_{p_j}\right)\exp\left(-\beta\frac{p^2\omega^2}{2}g_{ij}(\phi_0^i)\phi_p^i\phi_{-p}^j + i\beta q\omega{\psi_{-p}^i}^*\left(g_{ij}(\phi_0^i)\psi_p^j\right) + \dots\right)\\ &= \prod_{p\neq 0}\int\left(\sqrt{\det(g)}\prod_{i=1}^n\frac{d\phi_p^i}{\sqrt{2\pi}}\right)\exp\left(-\beta\frac{p^2\omega^2}{2}g_{ij}(\phi_0^i)\phi_p^i\phi_{-p}^j\right)\times\int\left(\prod_{j=1}^n d{\psi_p^j}^* d\psi_{p_j}\right)\exp\left(i\beta q\omega{\psi_{-p}^i}^*\psi_{p_j}\right)\end{aligned}$$

Diagonalising g_{ij} by an orthogonal matrix and rescaling ψ, by $\beta^{\frac{1}{4}}$ we have

$$\mathcal{I}_1 = \left(\prod_{p\neq 0}\sqrt{\frac{\det(g)}{(2\pi)^n}}\cdot\sqrt{\frac{(\pi)^n}{(\frac{1}{2}\beta p^2\omega^2)^n\det(g)}}\right)\left(\prod_{p\neq 0}(i\sqrt{\beta}p\omega)^n\right) = \mathbf{1}$$

The i in the numerator cancels because for every pair of integers $\frac{-ip\times ip}{\sqrt{p^2\times(-p)^2}} = 1$

Now the zero mode integral becomes:

$$\begin{aligned}\mathcal{I}_0 &= \int\left(\sqrt{\frac{\det(g)}{(2\pi)^n}}\prod_{i=1}^n d\phi_0^i\right)\left(\prod_{j=1}^n d{\psi_0^j}^* d\psi_{0_j}\right)\exp\left(-\frac{1}{4}R_{ijkl}{\psi_0^i}^*{\psi_0^j}^*\psi_0^k\psi_0^l\right)\\ &= \int\left(\sqrt{\frac{\det(g)}{(2\pi)^n}}\prod_{i=1}^n d\phi_0^i\right)\left(\prod_{j=1}^n d{\psi_0^j}^* d\psi_{0_j}\right)\exp\left(-\frac{1}{4}{R_{ij}}^{kl}{\psi_0^i}^*{\psi_0^j}^*\psi_{0k}\psi_{0l}\right)\end{aligned}$$

It is trivial to see that $\mathcal{I}_0 = 0$ for odd dimensions, $n = 2m+1$, because the Grassman integral would be unsaturated by any term in the expansion of $\exp\left(-\frac{1}{4}R_{ijkl}{\psi_0^i}^*{\psi_0^j}^*\psi_0^k\psi_0^l\right)$, since all these terms would have an even number of ψ's & ψ^*'s. So, $\Delta = 0$ for $n = 2m+1$ dimensional Manifolds. For even dimensional manifolds, this becomes:

$$\begin{aligned}\Delta = \mathcal{I}_0 &= \frac{1}{(\pi)^{n/2}}\int\underbrace{\left(\sqrt{\det(g)}\prod_{i=1}^n d\phi_0^i\right)}_{\text{Volume element of manifold}}\left(\prod_{j=1}^n d{\psi_0^j}^* d\psi_{0_j}\right)\exp\left(-\frac{1}{4}{R_{ij}}^{kl}{\psi_0^i}^*{\psi_0^j}^*\psi_{0k}\psi_{0l}\right)\\ &= \frac{1}{(\pi)^m}\int d(\mathrm{Vol})\int d{\psi_0^1}^* d\psi_{01}d{\psi_0^2}^* d\psi_{02}\dots d{\psi_0^{2m}}^* d\psi_{02m}\frac{\left(-\frac{1}{4}{R_{ij}}^{kl}{\psi_0^i}^*{\psi_0^j}^*\psi_{0k}\psi_{0l}\right)^m}{m!}\\ &= \frac{(-1)^m}{4^m m!\pi^m}\int d(\mathrm{Vol})\int d{\psi_0^1}^* d\psi_{01}d{\psi_0^2}^* d\psi_{02}\dots d{\psi_0^{2m}}^* d\psi_{02m}\left({R_{ij}}^{kl}{\psi_0^i}^*\psi_{0l}{\psi_0^j}^*\psi_{0k}\right)^m\\ &= \frac{(-1)^m}{4^m m!\pi^m}\int d(\mathrm{Vol})\int d{\psi_0^1}^* d\psi_{01}\dots\left({R_{i_1j_1}}^{k_1l_1}{\psi_0^{i_1}}^*\psi_{0l_1}{\psi_0^{j_1}}^*\psi_{0k_1}\right)\dots\left({R_{i_mj_m}}^{k_ml_m}{\psi_0^{i_m}}^*\psi_{0l_m}{\psi_0^{j_m}}^*\psi_{0k_m}\right)\end{aligned}$$

We can easily see that only terms with i_p,j_p distinct & k_p,l_p distinct contribute & transpositions give a relative minus sign. The overall sign of the expression is fixed by looking at the sign of, say, ${R_{12}}^{12}\ldots {R_{(2m-1),(2m)}}^{(2m-1),(2m)}$

$$\therefore \Delta = \frac{(-1)^m}{2^{2m}m!\pi^m}\int d(\mathrm{Vol})\epsilon^{i_1j_1\ldots i_mj_m}\epsilon_{k_1l_1\ldots k_ml_m}{R_{i_1j_1}}^{k_1l_1}\ldots {R_{i_mj_m}}^{k_ml_m}$$

$$\Rightarrow Tr(-1)^{\hat{F}}e^{-\beta\hat{\mathcal{H}}} = \chi(M),\ \text{the Euler characteristic of } M,\ \dim(M) = n = 2m$$

This relation between the index of the operator d & a topological property of the manifold (the Euler characteristic in this case) on which it is operating, is acting, is a special case of the celebrated Atiyah-Singer index theorem [15]. The statement of the theorem is that *the topological index*[8] *and the analytical index of an operator are identical.*

4 Conclusion

In this report we outlined a proof of a specific instance of the Atiyah-Singer index theorem, using Supersymmetry. This proof involving a path integral, while, perhaps lacking in mathematical rigour certainly gives us an interesting way to look at the mathematical theorem using insights from the Physical model that we have employed. Hopefully such interfaces, between theoretical Physics and mathematics will prove beneficial for both the disciplines and result in the kind of synthesis talked about by Michael Atiyah et al in [17]

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Appendix A

The Path Integral

The topics covered in this appendix are far better dealt with in several books & review articles (such as [11], [13] or [14]). The reason for placing this appendix here is to make this report self contained and accessible to fellow students like myself, as well as a place to collect my thoughts and clarify my understanding of these tools & techniques of theoretical physics.

Consider a quantum mechanical system, with position $\hat{q}$, & momentum $\hat{p}$, satisfying the commutation relation (setting $\hbar = 1$):

$$[\hat{q}, \hat{p}] = i$$

Then, we may be interested in the *propagator* $K(q_i, q_f, T)$:

$$K(q_i, q_f, T) = \langle q_f|\exp{(-i\hat{\mathcal{H}}T)}|q_i\rangle = \langle q_f|\exp{(-i\hat{\mathcal{H}}\epsilon)}\cdots\exp{(-i\hat{\mathcal{H}}\epsilon)}|q_i\rangle$$

Now inserting a complete set of states $\int dq_k|q_k\rangle\langle q_k|$ in between the infinitesimal time evolutions,

[8] In the case of the *de Rham* complex, this simplifies to $\chi(M)$, the Euler characteristics, which we showed was equal to the index of d.

the propagator becomes:

$$
\begin{aligned}
& \int \left(\prod_{1}^{n-1} dq_k \right) \langle q_f | \exp{(-i\hat{\mathcal{H}}\epsilon)} | q_1 \rangle \langle q_1 | \exp{(-i\hat{\mathcal{H}}\epsilon)} | q_2 \rangle \langle q_2 | \cdots | q_{n-1} \rangle \langle q_{n-1} | \exp{(-i\hat{\mathcal{H}}\epsilon)} | q_i \rangle \\
= & \int \left(\prod_{1}^{n-1} dq_k \right) \left(\prod_{0}^{n-1} dp_j \right) \langle q_f | p_0 \rangle \langle p_0 | \exp{(-i\hat{\mathcal{H}}\epsilon)} | q_1 \rangle \langle q_1 | p_1 \rangle \cdots \langle q_{n-1} | p_{n-1} \rangle \langle p_{n-1} | \exp{(-i\hat{\mathcal{H}}\epsilon)} | q_i \rangle \\
= & \int \left(\prod_{1}^{n-1} dq_k \right) \left(\prod_{0}^{n-1} \frac{dp_j}{2\pi} \right) \times \exp \left(-i \sum_{m=0}^{n} [\mathcal{H}(p_m, q_{m+1})\epsilon - p_m(q_m - q_{m+1})] \right) \\
\approx & \int [dpdq] \exp \left(i \int_0^T dt [p\dot{q} - \mathcal{H}(p,q)] \right)
\end{aligned}
$$

Now the Hamiltonian's dependence on p is usually quadratic, in which case the gaussian integral over the momenta can be easily evaluated, giving:

$$
K(q_i, q_f, T) = \int [dq] exp \left(i \int_0^T dt \mathcal{L}(q, \dot{q}) \right)
$$

The trace of $exp(-i\hat{\mathcal{H}}T)$ can also be evaluated by setting $q_i = q_f = q'$ and integrating over all q', thus giving the path integral with periodic boundary conditions (PBC) on q

$$
Tr \exp{(-i\hat{\mathcal{H}}T)} = \int_{PBC} [dq] exp \left(i \int_0^T dt \mathcal{L}(q, \dot{q}) \right)
$$

This function can be analytically continued to $it \to u$, and time T can be replaced by inverse temperature β, giving the partition function:

$$
Z = Tr \exp{(-\beta\hat{\mathcal{H}})} = \int_{PBC} [dq] exp \left(\int_0^\beta du \mathcal{L}(q, \dot{q}) \right) = \int_{PBC} [dq] exp \left(- \int_0^\beta du \mathcal{L}_E(q, \dot{q}) \right)
$$

All this can essentially be generalised to the case of a bosonic field, ϕ, but the case of fermions is slightly different.

Fermions

Working with the Grassman variables, ψ & ψ^* we get a similar expression for the path integral except that the boundary conditions are antiperiodic. We can easily see this as follows:

$$
\begin{aligned}
Z(\beta) &= \sum_n \langle n | \exp{(-\beta\hat{\mathcal{H}})} | n \rangle = \sum_n \int d\psi^* d\psi e^{-\psi^*\psi} \langle n | \psi \rangle \langle \psi | \exp{(-\beta\hat{\mathcal{H}})} | n \rangle \\
&= \sum_n \int d\psi^* d\psi (1 - \psi^*\psi) \langle n | (|0\rangle + |1\rangle\psi)(\langle 0| + \psi^* \langle 1|) \exp{(-\beta\hat{\mathcal{H}})} | n \rangle \\
&= \sum_n \int d\psi^* d\psi (1 - \psi^*\psi) [\langle 0 | \exp{(-\beta\hat{\mathcal{H}})} | n \rangle \langle n | 0 \rangle - \psi^*\psi \langle 1 | \exp{(-\beta\hat{\mathcal{H}})} | n \rangle \langle n | 1 \rangle + \\
& \quad \psi \langle 0 | \exp{(-\beta\hat{\mathcal{H}})} | n \rangle \langle n | 1 \rangle + \psi^* \langle 1 | \exp{(-\beta\hat{\mathcal{H}})} | n \rangle \langle n | 0 \rangle] \\
&= \sum_n \int d\psi^* d\psi (1 - \psi^*\psi) [\langle 0 | \exp{(-\beta\hat{\mathcal{H}})} | n \rangle \langle n | 0 \rangle - \psi^*\psi \langle 1 | \exp{(-\beta\hat{\mathcal{H}})} | n \rangle \langle n | 1 \rangle + \\
& \quad \psi \langle 0 | \exp{(-\beta\hat{\mathcal{H}})} | n \rangle \langle n | 1 \rangle - \psi^* \langle 1 | \exp{(-\beta\hat{\mathcal{H}})} | n \rangle \langle n | 0 \rangle] \\
&= \int d\psi^* d\psi (1 - \psi^*\psi) \langle -\psi | \exp{(-\beta\hat{\mathcal{H}})} | \psi \rangle \\
&= \int d\psi^* d\psi e^{-\psi^*\psi} \langle -\psi | \exp{(-\beta\hat{\mathcal{H}})} | \psi \rangle
\end{aligned}
$$

The last propagator from $-\psi \to \psi$ can be again split up into time slices and rewritten just like the bosonic integral. The only difference is that the boundary conditions are changed from periodic to antiperiodic. This is the expression for the fermionic path integral, but interestingly, when calculating the Witten index, Δ, we have a $(-1)^{\hat{F}}$ which contributes an additional - sign, thus making the boundary conditions periodic again.

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